

# Open String Thermodynamics and D-Branes

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We study the thermodynamics of open superstrings in the presence of  $p$ -dimensional D-branes. We get some finite temperature dualities relating the one-loop canonical free energy of open strings to the self-energy of D-branes at dual temperature. For the open bosonic string the inverse dual temperature is, as expected, the dual length under T-duality,  $4\pi^2\alpha'/\beta$ . On the contrary, for the  $SO(N)$ , type-I superstring the dual temperature is given by  $\beta$ -duality,  $2\pi^2\alpha'/\beta$ . We also study the emergence of the Hagedorn singularity in the dual description as triggered by the coupling of the D-brane to unphysical tachyons as well as the high temperature limit.

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## 1. Introduction

Apart from a few exceptions [1], open superstring theory has been left in a state of semi-oblivion during the last ten years mainly due to the good phenomenological prospects of closed superstring models. However, after the avalanche of results following the *second superstring revolution* in 1995 it has become clear that open strings play an important rôle in the non-perturbative dynamics of closed string models. At this moment there are solid evidences that the weakly coupled  $SO(32)$ , type-I superstring describes the strong coupling regime of the  $SO(32)$  heterotic string [2]. Dirichlet open string theory [3] has also been incorporated into this landscape after the result of Polchinski [4] that D-branes [5] carry R-R charge and therefore are candidates for the solitonic states demanded by S-duality of the type-IIB superstring [2].

Open superstrings at finite temperature is an appealing scenario which should be re-explored in the light of the new results [6][7][8]. One interesting issue to study is the physics of strings at the Hagedorn temperature. Since the Hagedorn divergence indicates a breakdown of the perturbative formalism at high temperatures it is possible that the brand new string dualities could be of any use in deciding whether this signals any kind of phase transition or a maximum temperature of the string ensemble. The main obstacle in addressing this problem is the breaking of supersymmetry at non-zero temperature; consequently the application of zero temperature dualities in this setup is highly delicate.

In the present note we are going to be concerned with another interesting problem in the subject of thermal strings: finite temperature duality [9][10]. It is well known that the partition function of the ten-dimensional heterotic string enjoys a  $\beta$ -duality<sup>2</sup> relation connecting low and high temperatures ( $\beta = T^{-1}$ )

$$Z_{\text{het}}(\beta) = Z_{\text{het}}\left(\frac{2\pi^2\alpha'}{\beta}\right).$$

It is important to stress that the  $\beta$ -duality of the heterotic string differ from the usual T-duality by a factor two. A consequence of this formal relation is the existence of a well-defined high-temperature phase (beyond the Hagedorn point) with *anomalous* thermodynamical properties [11][10]. There are, however, no such self-duality for the type-II [12] and type-I superstrings.

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<sup>2</sup> In this paper I will use the term  $\beta$ -duality to label the transformation  $R \rightarrow \alpha'/2R$  in opposition to T-duality,  $R \rightarrow \alpha'/R$ , with  $R$  the compactification radius.

In type-I models strings and D-branes play a dual rôle under T-duality [13]; therefore it is likely for a dual temperature description of a gas of open strings to involve D-branes. Of course this has to be true for the bosonic string, since in this case finite temperature duality is just T-duality. For open fermionic strings, on the other hand, the boundary conditions of the space-time fermions makes the situation more involved and one must proceed with some care. In the next section we will study the canonical free energy of open superstrings and D-branes and, going to the closed string channel, will obtain some duality relations between the gas of strings and a number of “hot” semiclassical D-branes. To make the exposition lighter, we will present in some detail the case of the bosonic string to center ourselves later in the much more interesting case of the  $SO(N)$  open superstring. In section 3 we will look at the Hagedorn transition from the D-brane perspective and the high temperature limit of open strings will be study. Finally, in section 4 we will summarize our conclusions.

## 2. Open strings and D-branes at finite temperature

Our first task will be to study a thermal ensemble of open strings in the presence of a  $p$ -dimensional D-brane. The best way to get the one-loop canonical free energy of such a system is to begin with a gas of open strings in a  $D - 1$  toroidal space<sup>3</sup> with radii  $R_i$  ( $i = 1, \dots, D - 1$ ) and then take  $R_1, \dots, R_p \rightarrow \infty$  at the same time than  $R_{p+1}, \dots, R_{D-1} \rightarrow 0$ . The euclidean time is compactified at a fixed length  $\beta = 2\pi R_0 = T^{-1}$  and target fermions are taken to be antiperiodic in this circle. The one-loop canonical free energy can be easily computed by summing over the individual contributions of all the fields in the string taking into account their boundary conditions [14][15]

$$\log Z(\beta) = -\text{Tr}_{\mathcal{H}} \left[ (-1)^F \int_0^\infty \frac{dt}{2t} e^{-2\pi\alpha' t(k^2 + M^2)} \right] + \text{counterterms},$$

where the trace is over the open string Hilbert space and  $k^2$ ,  $M^2$  are respectively the momentum and mass operators and  $F$  is the target space fermion number. The counterterms remove the vacuum energy in the zero-temperature limit  $\beta \rightarrow \infty$  whenever it

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<sup>3</sup>  $D = 10$  for the superstring and  $D = 26$  for the bosonic string.

is non-vanishing. Proceeding as described above we find for the bosonic string without Chan-Paton factors<sup>4</sup>

$$\log Z_p(\beta) = -\beta \int_0^\infty \frac{dt}{2t} (8\pi^2 \alpha' t)^{-\frac{p+1}{2}} [\eta(it)]^{-24} \left[ \theta_3 \left( 0 \left| \frac{i\beta^2}{8\pi^2 \alpha' t} \right. \right) - 1 \right]. \quad (2.1)$$

In the case of the  $SO(N)$ , type-I superstring the result is

$$\begin{aligned} \log Z_p(\beta) = & -\frac{N^2}{2} \int_0^\infty \frac{dt}{2t} (8\pi^2 \alpha' t)^{-\frac{p}{2}} \frac{\theta_2^4(0|it)}{2\eta^{12}(it)} \theta_4 \left( 0 \left| \frac{2i\pi^2 \alpha' t}{\beta^2} \right. \right) \\ & + \frac{N}{2} \int_0^\infty \frac{dt}{2t} (8\pi^2 \alpha' t)^{-\frac{p}{2}} \frac{\theta_2^4(0|it + 1/2)}{2\eta^{12}(it + 1/2)} \theta_4 \left( 0 \left| \frac{2i\pi^2 \alpha' t}{\beta^2} \right. \right). \end{aligned} \quad (2.2)$$

The first term represents the contribution of the annulus and the second one that of the Möbius strip. In both cases  $(1/\beta) \log Z_p(\beta)$  can be interpreted as the canonical free energy of a gas of open strings living in the  $(p+1)$ -dimensional world-volume of the D-brane in which their endpoints live. Nevertheless, since open strings can fluctuate in the directions transverse to the D-brane they probe the full  $D$ -dimensional space as it must be since we are dealing with a critical string. Actually, it is interesting to notice that the only way to get a *purely*  $(p+1)$ -dimensional string theory would be to take the D-brane tension  $T_p$  to infinity while freezing the string tension at the same time. However this is not possible, since  $T_p$  scales as  $T_p \sim (\alpha')^{-p+(D-4)/2}$ . Thus we cannot take  $T_p \rightarrow \infty$  without at the same time taking  $\alpha'$  either to zero or to infinity (in the marginal case  $p = (D-4)/2$  the D-brane tension is a fixed numerical constant).

Up to now we have used the open string point of view. It is interesting, however, to look at the same problem from the D-brane perspective. What we have now, instead, is a D-brane located at a fixed point of the space which couples to closed strings. At zero temperature the vacuum energy can be viewed as due to the emission and absorption of (bosonic) closed string states. For the bosonic string this amounts to a divergent result, while for superstrings, on the contrary, the total vacuum energy is zero due to the cancellation between the contributions of R-R and NS-NS states. This is viewed in the open string channel as the result of the vanishing of the one-loop cosmological constant for the type-I superstring.

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<sup>4</sup> In what follows we will be rather cavalier in dealing with the standard divergences in the bosonic string amplitudes since they are a mere artifact of the tachyonic ground state. In any case these kind of divergences are irrelevant for most of our analysis.

At non-zero temperature supersymmetry is broken in the open string channel of the supersymmetric string and therefore there is no non-renormalization theorem which prevents us from having a net vacuum energy. Going to the closed string channel this means that the contributions to the self-energy of the D-brane coming from R-R and NS-NS states do not cancel any more and thus they are weighted differently in the sum over closed string states. In what follows we will construct the prescription to compute this “thermal” self-energy of the D-brane.

In order to illustrate the techniques to be used here we will work out first in some detail the case of the oriented bosonic string, in which we already know the kind of results to be expected. We can rewrite (2.1) in the closed string channel by changing to a new integration variable  $s = t^{-1}$  (for the time being we will include the divergent vacuum energy)

$$Z_p(\beta) = -\frac{1}{2} \beta (8\pi^2 \alpha')^{-\frac{p+1}{2}} \int_0^\infty ds s^{-\frac{25-p}{2}} [\eta(is)]^{-24} \theta_3 \left( 0 \left| \frac{i\beta^2 s}{8\pi^2 \alpha'} \right. \right).$$

Now we use the expansion of the Dedekind eta function

$$[\eta(is)]^{-24} = \sum_{N=0}^{\infty} c(N) e^{-2\pi s(N-1)} \quad c(N) \in \mathbf{Z}$$

and after trading variables again in the integral,  $s \rightarrow s' = \pi \alpha' s/2$ , we have

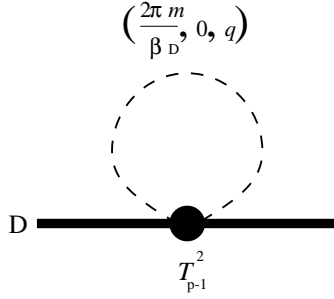
$$Z_p(\beta) = -\frac{1}{8} (4\pi)^{\frac{p-25}{2}} \left[ \frac{\sqrt{\pi}}{16} (4\pi^2 \alpha')^{\frac{11-p}{2}} \right]^2 \sum_{N=0}^{\infty} c(N) \sum_{m \in \mathbf{Z}} \int_0^\infty ds s^{-\frac{25-p}{2}} e^{-s \left[ 4\pi^2 m^2 \beta^2 + \frac{4}{\alpha'} (N-1) \right]};$$

the term between square brackets gives the  $p$ -brane tension; in the exponent it is easy to recognize the mass formula for the closed string with  $N$  the oscillator level of the right-moving modes. After trivial manipulations and taking into account that  $T_{p-1} = T_p (4\pi^2 \alpha')^{1/2}$  we can recast  $Z_p(\beta)$  in the more suggestive form

$$\begin{aligned} Z_p(\beta) &= -\frac{1}{8} T_{p-1}^2 \sum_i \left[ \frac{\beta}{4\pi^2 \alpha'} \sum_{m \in \mathbf{Z}} \int \frac{d^{25-p} q}{(2\pi)^{25-p}} \right] \int_0^\infty ds e^{-s \left[ q^2 + \frac{\beta^2 m^2}{4\pi^2 \alpha'^2} + M_i^2 \right]} \\ &= -\frac{1}{8} T_{p-1}^2 \sum_i \left[ \frac{\beta}{4\pi^2 \alpha'} \sum_{m \in \mathbf{Z}} \int \frac{d^{25-p} q}{(2\pi)^{25-p}} \right] \Delta_i \end{aligned} \tag{2.3}$$

where  $\Delta_i = (q^2 + M_i^2 + \beta^2 m^2 / 4\pi^2 \alpha'^2)^{-1}$  is the propagator for the  $i$ -th state in the closed bosonic string spectrum. Momentum conservation in the directions with Neumann boundary conditions sets this momentum to zero. On the other hand, momentum is not conserved

in the directions with Dirichlet boundary conditions so we integrate over the momenta transverse to the D-brane. The interesting point is that the term  $\beta^2 m^2 / 4\pi^2 \alpha'^2$  can be interpreted as the discrete momentum squared for a closed string in a circle with length  $\beta_D = 4\pi^2 \alpha' / \beta$  [16]. The compactified coordinate has now Dirichlet boundary conditions, as it is inferred from the non-conservation of the discrete momentum in that direction. Therefore the physical picture is that of a  $(p-1)$ -dimensional D-brane<sup>5</sup> emitting and absorbing closed strings at inverse temperature  $4\pi^2 \alpha' / \beta$ . Actually, we can read off from (2.3) the thermal self-energy for a static semiclassical Dirichlet  $(p-1)$ -brane (fig. 1)



**Fig. 1:** Effective Feynman diagram for the self-energy of the bosonic D-brane.

$$\Pi_{p-1}(\beta) = -T_{p-1}^2 \sum_i \left[ \frac{\beta}{4\pi^2 \alpha'} \sum_{m \in \mathbf{Z}} \int \frac{d^{25-p} q}{(2\pi)^{25-p}} \right] \int_0^\infty ds e^{-s \left[ q^2 + \frac{\beta^2 m^2}{4\pi^2 \alpha'^2} + M_i^2 \right]};$$

therefore we arrive at the following *duality* relation between the Helmholtz free energy of the open bosonic string and the D-brane self-energy

$$F_p(\beta) = \frac{1}{8\beta} \Pi_{p-1} \left( \frac{4\pi^2 \alpha'}{\beta} \right). \quad (2.4)$$

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<sup>5</sup> If we regard this as a purely euclidean theory this is the correct picture. However if we think of our euclidean theory as the analytic continuation of a Minkowskian string theory the fact that the time coordinate has Dirichlet boundary conditions means that the  $p$ -brane is localized in time, although it can be extended in space if  $p > 0$ . As a consequence what we have is a  $p$ -brane whose world-volume is the  $p$ -brane itself.

Notice the appearance of  $\beta^{-1}$  in the prefactor instead of the  $\beta^{-2}$  typical of the T-duality of closed bosonic strings. As we will see later on this is linked to the different  $\beta \rightarrow 0$  behavior of open and closed strings [10].

Before leaving the bosonic string let us remark that expression (2.1) admits a D-brane interpretation directly in the open string channel. By applying a Poisson resummation on the thermal theta function, and using the expressions of reference [5], one realizes that  $\beta F_p(\beta)$  is equal to the energy of a periodic array of  $(p-1)$ -dimensional D-branes separated by a distance  $\beta_D = 4\pi^2\alpha'/\beta$ . Actually, this is nothing more than a trivial application of the relation between open strings and D-branes under T-duality, since by performing a T-duality on the compactified dimension we have now Dirichlet boundary conditions in that coordinate, increasing the number of transverse directions to the D-brane and reducing consequently its dimension from  $p$  to  $p-1$ . This is consistent with our interpretation in the closed string channel.

The picture found for the bosonic string is not much of a surprise, since in that case finite temperature amounts to a simple toroidal compactification. The study of the  $SO(N)$ , type-I superstring is by far richer since now target fermions in the open string channel are antiperiodic along the compactified dimension. As we did in the bosonic case, we begin by transforming (2.2) into the closed string channel. To proceed, however, it is much more convenient to rewrite  $\log Z_p(\beta)$  explicitly as the vacuum energy of a toroidal compactification modded out by the operator  $\alpha = e^{iP_0\beta - 2\pi i J_{12}}$  [17], where  $P_0$  is the discrete momentum in the euclidean time direction and  $J_{12}$  is the generator of rotations in the  $X^1$ - $X^2$  plane. The result for the annulus part is<sup>6</sup>

$$\begin{aligned} \log Z_A(\beta) = & -\frac{N^2}{8} \int_0^\infty \frac{dt}{2t} (8\pi^2\alpha' t)^{-\frac{p}{2}} [\eta(it)]^{-12} \left\{ [\theta_3^4(0|it) - \theta_4^4(0|it) - \theta_2^4(0|it)] \theta_3 \left( 0 \left| \frac{2i\pi^2\alpha' t}{\beta^2} \right. \right) \right. \\ & \left. + [\theta_3^4(0|it) - \theta_4^4(0|it) + \theta_2^4(0|it)] \theta_4 \left( 0 \left| \frac{2i\pi^2\alpha' t}{\beta^2} \right. \right) \right\} \end{aligned} \quad (2.5)$$

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<sup>6</sup> A related study of the annulus partition function for the open superstring with  $p=0$  can be found in [7]. I thank M.B. Green for attracting my attention to this reference.

The term multiplying the *thermal* theta-function  $\theta_3$  vanishes identically due to the well-known *aequatio identica satis abstrusa*. In the Möbius amplitude we find

$$\begin{aligned} \log Z_M(\beta) = & \frac{N}{8} \int_0^\infty \frac{dt}{2t} (8\pi^2 \alpha' t)^{-\frac{p}{2}} [\eta(it + 1/2)]^{-12} \\ & \times \left\{ [\theta_3^4(0|it + 1/2) - \theta_4^4(0|it + 1/2) - \theta_2^4(0|it + 1/2)] \theta_3 \left( 0 \left| \frac{2i\pi^2 \alpha' t}{\beta^2} \right. \right) \right. \\ & \left. + [\theta_3^4(0|it + 1/2) - \theta_4^4(0|it + 1/2) + \theta_2^4(0|it + 1/2)] \theta_4 \left( 0 \left| \frac{2i\pi^2 \alpha' t}{\beta^2} \right. \right) \right\}. \end{aligned} \quad (2.6)$$

Let us study first the orientable part of the amplitude. As we did in the bosonic case, we rewrite (2.5) in the closed string channel by changing variables  $s = t^{-1}$ . After this we can identify the different sectors in the new channel;  $\theta_3^4(0|is) - \theta_4^4(0|is)$  corresponds to closed string states in the NS-NS sector with  $G = G_L = G_R = +1$ , whereas  $\theta_2^4(0|is)$  represents the contribution of R-R states with  $G = G_L = \pm G_R = +1$ ;  $G$  is the  $G$ -parity of the closed string state and  $\pm G_L$  for the R-R states applies respectively when  $p$  is even or odd<sup>7</sup>. In the same manner,  $\theta_3^4(0|is) + \theta_4^4(0|is)$  and  $\theta_2^4(0|is)$  in the second part of the expression correspond respectively to NS-NS and R-R with  $G = -1$ . These latter are the states absent in the supersymmetric closed string theory and their contributions to the partition function disappear in the zero temperature limit. By rescaling the integration variable and identifying the mass formulae in the different sectors we arrive at

$$\begin{aligned} \log Z_A(\beta) = & -\frac{1}{16} \frac{N^2}{2} T_{p-1}^2 \left[ \frac{\beta}{2\pi^2 \alpha'} \sum_{m \in \mathbf{Z}} \int \frac{d^{9-p} q}{(2\pi)^{9-p}} \right] \int_0^\infty ds \left[ \sum_{(\text{NS}, +)} e^{-s \left[ q^2 + \frac{m^2 \beta^2}{\pi^2 \alpha'^2} + M_i^2 \right]} \right. \\ & \left. - \sum_{(\text{R}, +)} e^{-s \left[ q^2 + \frac{m^2 \beta^2}{\pi^2 \alpha'^2} + M_i^2 \right]} + \sum_{(\text{NS}, -)} e^{-s \left[ q^2 + \frac{(m+1/2)^2 \beta^2}{\pi^2 \alpha'^2} + M_i^2 \right]} - \sum_{(\text{R}, -)} e^{-s \left[ q^2 + \frac{(m+1/2)^2 \beta^2}{\pi^2 \alpha'^2} + M_i^2 \right]} \right]. \end{aligned} \quad (2.7)$$

The sums inside the integral correspond to the four types of closed string states, NS-NS and R-R with positive and negative  $G$ -parities. As it is the case in at zero temperature the contributions from NS-NS and R-R have opposite signs but now they cancel each other only in the  $G = +1$  sector. In fact, a very interesting picture for the orientable part of

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<sup>7</sup> This is because type-IIB closed strings couple to odd-dimensional D-branes while those of the type-IIA do to even-dimensional ones. We will see in a moment that in the closed string channel we deal with  $(p-1)$ -dimensional D-branes.



the “hot” D-brane emerges from (2.7). Because of the form of the discrete contributions to the propagator the closed strings can be thought to be living in a circle with length  $\beta_D = 2\pi^2\alpha'/\beta$ . As in the bosonic string, now  $X^0$  has Dirichlet boundary conditions, momentum is not conserved in that direction and we deal with a  $(p-1)$ -dimensional D-brane. Moreover, odd  $G$ -parity states have half-integer momentum numbers so they are treated as antiperiodic in the compactified dimension, or in other words, they behave as fermions from the thermal point of view [16]. Since the  $G$  parity operator is essentially the world-sheet fermion number the final conclusion is that in going from the open string to the D-brane picture we are, in a sense, trading the rôles of target and world-sheet statistics in the finite temperature setup. As in the case of the bosonic string one read from (2.7) the corresponding Feynman diagram in the effective D-brane theory and find that it can also be interpreted as the self-energy of the system of  $N$  D-branes where odd  $G$ -parity bosonic states are twisted (they are treated as fermions). This is equivalent to mod out the tree level closed string spectrum by the operator  $\tilde{\alpha} = e^{i\tilde{P}_0\beta_D}G$ , with  $\tilde{P}_0$  the momentum in the dual circle with length  $\beta_D$ .

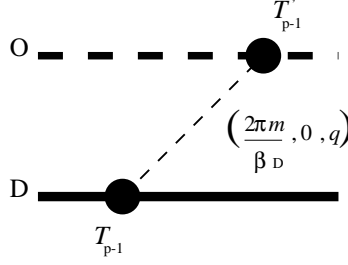
One remarkable thing about this picture is the emergence of the dual temperature under  $\beta$ -duality as the “temperature” of the Dirichlet  $(p-1)$ -brane, differing from the value implied by T-duality by a factor two. Then the annulus canonical free energy  $F_A(\beta)$  can be written in terms of the self-energy of the system of coincident D-branes at temperature  $2\pi^2\alpha'/\beta$

$$F_A(\beta) = \frac{1}{16\beta} \Pi_{p-1} \left( \frac{2\pi^2\alpha'}{\beta} \right).$$

Let us move to the Möbius amplitude. Now the closed string channel will describe a closed string emitted by the D-brane which ends in a crosscap located in the vicinity of the orientifold plane. Therefore what we expect to obtain is a description of the Möbius free energy in terms of the self-energy of the system D-brane-orientifold (fig. 2).

In order to go from the open to the closed string channel it is useful to re-express (2.6) in terms of modular functions depending on  $2t$ . The closed string channel is gotten then by the replacement  $2t \rightarrow (2s)^{-1}$ . Skipping the details we get

$$\begin{aligned} \beta F_M(\beta) = & \frac{N}{8} 2^{\frac{p-7}{2}} (4\pi^2\alpha')^{-\frac{p+1}{2}} \beta \int_0^\infty ds s^{-\frac{9-p}{2}} [\eta(is+1/2)]^{-12} \\ & \times \left\{ [\theta_3^4(0|is+1/2) - \theta_4^4(0|is+1/2) - \theta_2^4(0|is+1/2)] \theta_3 \left( 0 \left| \frac{2i\beta^2 s}{\pi^2\alpha'} \right. \right) \right. \\ & \left. + [\theta_3^4(0|is+1/2) - \theta_4^4(0|is+1/2) + \theta_2^4(0|is+1/2)] \theta_2 \left( 0 \left| \frac{2i\beta^2 s}{\pi^2\alpha'} \right. \right) \right\}. \end{aligned}$$



**Fig. 2:** Feynman diagram for the self-energy of the system D-brane-orientifold.

The identification of R-R and NS-NS states now is a little more involved. From [18] we learn that  $\theta_3^4 - \theta_4^4$  corresponds to R-R, while  $\theta_2^4$  is the contribution of NS-NS states, both with  $G = +1$ . Taking this into account and using some well-known properties of the Jacobi theta-functions [19] one arrives, after a redefinition of the proper time, at

$$\begin{aligned} \log Z_M(\beta) = & -\frac{1}{2} \frac{N}{2} T_{p-1} T'_{p-1} \left[ \frac{\beta}{2\pi^2 \alpha'} \sum_{m \in \mathbf{Z}} \int \frac{d^{9-p} q}{(2\pi)^{9-p}} \right] \\ & \times \int_0^\infty ds \left[ \sum_{(\text{NS}, +)} (-1)^{m+N_i} e^{-s \left[ q^2 + \frac{m^2 \beta^2}{\pi^2 \alpha'^2} + M_i^2 \right]} - \sum_{(\text{R}, +)} (-1)^{N_i} e^{-s \left[ q^2 + \frac{m^2 \beta^2}{\pi^2 \alpha'^2} + M_i^2 \right]} \right] \end{aligned} \quad (2.8)$$

where  $N_i$  is the total right-moving oscillator number and we have identified the tensions of the D-brane and the orbifold plane,  $T_p$  and  $T'_p$ . First of all we find that, in contrast with the annulus case, in the Möbius partition function we have contributions only from the even  $G$ -parity bosonic closed string states. In addition, from inspection of the exponents, we see that their discrete momenta correspond to those of closed strings living in a circle with length  $\beta_D = 2\pi^2 \alpha' / \beta$ , again the  $\beta$ -duality value that we also got for the annulus. It is interesting to notice that at finite temperature the cancellation between the R-R and NS-NS states only takes place for those states with even momentum number, whereas for odd  $m$  both contributions add up to give a net non-vanishing result. The general structure of (2.8) is

$$\log Z_M(\beta) = -\frac{1}{2} \frac{N}{2} T_{p-1} T'_{p-1} \sum_i \left[ \frac{\beta}{2\pi^2 \alpha'} \sum_{m \in \mathbf{Z}} \int \frac{d^{9-p} q}{(2\pi)^{9-p}} \right] \Omega_i \Delta_i$$

where again  $\Delta_i$  is the propagator for the  $i$ -th state of the type-II superstring.  $\Omega_i$  is a “twist” operator in the closed string channel that turns out to be

$$\Omega_i = \frac{1 + G_i}{2} e^{i \tilde{P}_0 \beta_D a} (-1)^{N_i}$$

with  $a = 1/2, 0$  for NS-NS and R-R states respectively; the projector selects the states with  $G = +1$ , the only ones coupling to the orientifold plane. This provides us with the “finite temperature” prescription for constructing the self-energy of the system  $(p - 1)$ -brane-orientifold, with the result

$$F_p(\beta) = \frac{1}{2\beta} \Pi_{p-1}^{\text{D-O}} \left( \frac{2\pi^2 \alpha'}{\beta} \right).$$

For the unoriented superstring an open string channel interpretation of (2.2) along the lines of the one offered for the bosonic string can also be made. In this case what we find is an array of  $(p - 1)$ -dimensional D-branes located at  $Y_D = 2\pi^2 \alpha' n / \beta$ , while the orientifold planes are at  $Y_O = \pi^2 \alpha' n / \beta$ , with  $n \in \mathbf{Z}$ . The annulus amplitude describes open strings stretched between two D-branes with a modding imposing that bosonic modes wrap an even number of times whereas the “winding” number for the fermionic ones is odd. For the Möbius amplitude the scenario is similar where now the endpoints of the string are located at a give D-brane and its mirror image with respect to the orientifold.

### 3. The high temperature limit

Once we have arrived at a D-brane description of open strings at finite temperature there are a couple of things that can be studied from the new point of view. The first one is the physical meaning of the Hagedorn temperature. As we already indicated in the introduction, this is a problem that one does not hope to solve in the D-brane scenario; however it is of some interest to see how the divergence arise in the dual picture.

In both the bosonic and the fermionic string the one-loop partition function in the closed string channel is written as the trace over closed string propagators at zero momentum in the spatial directions with Neumann boundary conditions. From (2.3), (2.7) and (2.8) we verify that divergences will arise whenever we have a tachyonic state for small  $\beta$ . Thinking in terms of D-branes, the Hagedorn divergence arises for small values of the dual “temperature” triggered by the self-interaction of the D-brane via a tachyon. This divergence is removed at high dual temperatures because then the effective mass squared of the closed string tachyons becomes positive. This is much more reminiscent of a field-theoretical scenario, in which tachyonic ground states are removed at high temperature by symmetry restoration [20]. In the annulus partition function there are tachyons in the

bosonic string and in the  $G = -1$  NS-NS sector of the type-I superstring. In the Möbius partition function, on the contrary, there are no tachyons since it only involves the  $G = +1$  sector of the type-II superstring, so its contribution is regular for all values of the temperature. In the open string channel the Hagedorn transition is marked by a winding open string becoming tachyonic at high temperatures [7].

Another interesting issue is the formal  $T \rightarrow \infty$  behavior. For the closed ten-dimensional superstrings the canonical free energy goes like  $\beta^{-2}\Lambda$  where  $\Lambda$  is a constant that in the case of the heterotic string coincides with the vacuum energy at  $T = 0$  [10]. This is the asymptotic behavior to be expected from a two-dimensional field theory and the conclusion is that heterotic strings have very few propagating states at high temperature. For the open bosonic string in the presence of a  $p$ -brane we can obtain the  $\beta \rightarrow \infty$  using general arguments. In section 2 we constructed  $F_p(\beta)$  by starting with the  $D$ -dimensional open string with all spatial dimensions compactified in circles and taking the limit  $R_i \rightarrow 0$  in those transverse to the D-brane world-volume. If we now also take the radius of the euclidean time to zero we will end up with the vacuum energy of open strings in the background of a  $(p - 1)$ -dimensional D-brane at zero temperature. This is exactly the result found by computing the limit explicitly<sup>8</sup> after regularizing the integral with an ultraviolet cutoff  $\epsilon$  (cf. [10])

$$F_p(\beta) \rightarrow -\frac{1}{\beta} \int_{\epsilon}^{\infty} \frac{dt}{2t} (8\pi^2 \alpha' t)^{-\frac{p}{2}} [\eta(it)]^{-24} = \frac{\Lambda_{\epsilon}}{\beta} \quad (3.1)$$

From the open string analog model point of view,  $\Lambda_{\epsilon}$  is nothing but the ultraviolet regularized vacuum energy of the whole collection of quantum fields in the open string attached to a  $(p - 1)$ -brane. Actually, it is easy to translate (3.1) to the closed string channel by going to (2.3) and replacing the discrete sum by an integral over a continuum momentum, so now we have a single  $[25 - (p - 1)]$ -dimensional integral. There, as expected, the divergence when  $\epsilon$  goes to zero is due to the closed string tachyon propagating in a long tube; in going to the closed string channel we are trading the ultraviolet cutoff  $\epsilon$  by an infrared one,  $1/\epsilon$ .

The situation with the superstring is complicated by funny boundary conditions. Since space-time fermions are antiperiodic in the timelike direction, in the limit  $\beta \rightarrow 0$  all fermionic degrees of freedom disappear and we are left only with space-time bosons.

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<sup>8</sup> I am indebted to M.A.R. Osorio for sharing with me his unpublished results on the high temperature limit of open superstrings.

This is true for both the annulus and the Möbius amplitude. In the D-brane picture the “temperature” goes to infinity and therefore the momentum in the time direction becomes continuum. So in equations (2.7) and (2.8) the sum over the integer is replaced by an integral over a momentum  $q_0$  and consequently, as with the bosonic string, we end up with an integral over a  $[9 - (p - 1)]$ -dimensional momentum. In the case of the annulus the integrand is the trace over the four bosonic sectors of  $\exp[-s(q^2 + M_i^2)]$ . In the Möbius amplitude, the alternate sign on the NS-NS sector makes this part to disappear in that limit so we are left only with the trace of the same exponential to the positive  $G$ -parity R-R states<sup>9</sup>. The corresponding  $\beta \rightarrow 0$  limits are

$$F_A(\beta) \rightarrow -\frac{1}{\beta} \frac{N^2}{4} \int_{\epsilon}^{\infty} \frac{dt}{2t} (8\pi^2 \alpha' t)^{-\frac{p}{2}} \frac{\theta_3^4(0|it) - \theta_4^4(0|it)}{\eta^{12}(it)}$$

$$F_M(\beta) \rightarrow \frac{1}{\beta} \frac{N}{4} \int_{\epsilon}^{\infty} \frac{dt}{2t} (8\pi^2 \alpha' t)^{-\frac{p}{2}} \frac{\theta_3^4(0|it + 1/2) - \theta_4^4(0|it + 1/2)}{\eta^{12}(it + 1/2)}$$

It is easy to realize that the coefficients of  $\beta^{-1}$ , once transformed to the closed string channel, can be interpreted as the modded “zero temperature” self-energy of the D-brane. The regularization of the integrals is needed in the case of the annulus amplitude, since we have a divergence associated with the closed string NS-NS tachyon. In the Möbius case, however, there is no tachyon in the closed string channel and therefore any infrared divergence can only be due to massless modes; indeed, the integral will be infrared finite in the closed string channel ( $t \rightarrow 0$ ) whenever  $p < 8$ . Ultraviolet convergence, on the other hand, is guaranteed by the tower of massive states.

This high temperature behavior might seem somewhat exstrange. After all, the absence of the modular group  $SL(2, \mathbf{Z})$  in the open string at one-loop is usually taken as an indication that open string theory is more or less equivalent to a collection of quantum fields. This being so, it seems natural that in the limit  $\beta \rightarrow 0$  we should have  $F_p(\beta) \sim \beta^{-(p+1)}$ . There is however a very important point to remember: the asymptotic behavior  $\beta^{-D}$  for the one-loop free energy of quantum fields in a  $D$ -dimensional space-time is gotten provided that there is no ultraviolet divergences in the free energy other than the ones already present in the zero-temperature theory. Although this condition is fulfilled individually for each field in the string spectrum this is not the case for the collection as a whole. In the string partition function we have the Hagedorn divergence triggered by the exponential

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<sup>9</sup> This is a consequence of the fact NS-NS states in the Möbius strip correspond in the open string channel to R target fermions.

growth of the number of states per mass level; this infinity appears in the ultraviolet region in the proper time representation of  $F_p(\beta)$ . It is necessary then to regularize the ultraviolet behavior of the integral in order to extract some sensible result when the temperature goes to infinity and therefore we have to pick up a regularization length scale  $\Lambda$ . The most natural choice is to relate this scale to the string scale by defining  $\Lambda^2 = \epsilon \alpha'$ . Then in the limit  $\beta \rightarrow 0$  we have

$$F_p(\beta) \sim \frac{1}{\beta \Lambda^p} \sum_i f(\Lambda M_i) + O(e^{-\frac{\Lambda^2}{\beta^2}})$$

where the sum is over an appropriate subset of the string spectrum. This is exactly what we have been doing in this section. Had we redefined the proper time  $t \rightarrow \beta^2 t / \alpha'$  in equations (2.1) and (2.2) and placed an ultraviolet cutoff  $\epsilon$  (the procedure which gives us the  $\beta^{-D}$  behavior for a single quantum field), the final result would be  $\beta^{-(p+1)} g(\beta^2 / \alpha')$  where now, however,  $g(x)$  has no well-defined expansion around  $x = 0$ .

#### 4. Conclusions

In this letter we have studied finite temperature dualities for open strings in the presence of  $p$ -dimensional D-branes. By writting the one-loop canonical free energy in the closed string channel we re-expressed  $F_p(\beta)$  for the bosonic string without Chan-Paton factors in terms of the self-energy of a semiclassical  $(p-1)$ -dimensional Dirichlet brane at inverse temperature  $4\pi^2 \alpha' / \beta$ . In the case of the type-I,  $SO(N)$  superstring the annulus free energy is given by the self-energy of a system of  $(p-1)$ -branes at  $\beta_D = 2\pi^2 \alpha' / \beta$ , where the rôle of the space-time statistic is played by the  $G$ -parity of the closed string state. For the Möbius part the self-energy of the system D-brane-orientifold receives contributions only from the even  $G$ -parity closed string states although they are affected by phases coming from the “twist” operator.

Remarkably, in both the annulus and the Möbius type-I amplitudes the dual temperature felt by the closed strings coupled to the D-brane is determined by  $\beta$ -duality,  $2\pi^2 \alpha' / \beta$ . It is very important to stress that, either working with  $\alpha'_{\text{open}}$  or  $\alpha'_{\text{closed}}$ , what characterizes  $\beta$ -duality is the factor one half relative to the corresponding dual length under T-duality. Although  $\beta$ -duality is well-known in the context of heterotic strings at finite temperature, in fact it has also appeared in non-heterotic contexts, for example in the formal temperature duality of type-II superstrings [10][12]. Its emergence also in the open superstring

scenario strongly suggest that  $\beta$ -duality has to be viewed as a correction of T-duality in toroidal compactifications of any fermionic string with antiperiodic target fermions. It would be interesting to clarify the physical origin of such correction.

We have also studied the emergence of the Hagedorn singularity in the dual picture and verified that it is triggered by the emission of tachyonic closed string states by the D-brane. The absence of this singularity in the Möbius amplitude steems from the absence of tachyonic interchanges between the D-brane and the orientifold plane. Studying the formal limit  $\beta \rightarrow 0$  we find that the one-loop canonical free energy diverges as  $\Lambda_\epsilon \beta^{-1}$  in contrast with the closed string behavior as  $\beta^{-2}$  [10]. The regularized constant  $\Lambda_\epsilon$  is interpreted in the closed string channel as the “zero temperature” self-energy of the D-brane coupling to NS-NS and R-R closed string states with  $G = \pm 1$  (or of the D-brane-orientifold system interchanging only R-R states with  $G = 1$  in the case of the Möbius amplitude). In the open string channel this same constant corresponds to the vacuum energy of open strings in the presence of a  $(p - 1)$ -dimensional D-brane with target fermions eliminated. This is just a stringy version of dimensional reduction at high temperature.

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